$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$$

UNILATERAL L.T.

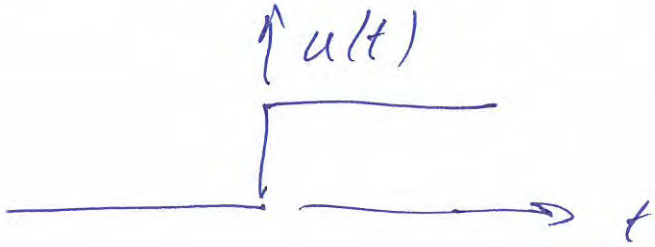
$$\mathcal{L}\{f(t)\} = F(s)$$

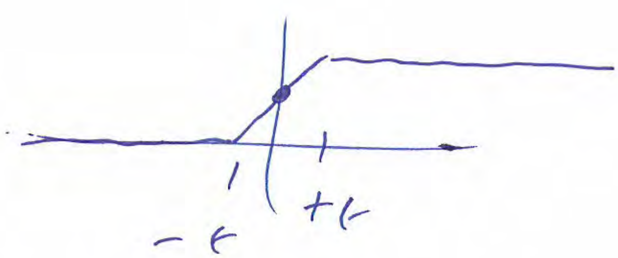
$$f(t) \longleftrightarrow F(s)$$

TIME DOMAIN — FREQUENCY DOMAIN

SINGULARITY FUNCTIONS

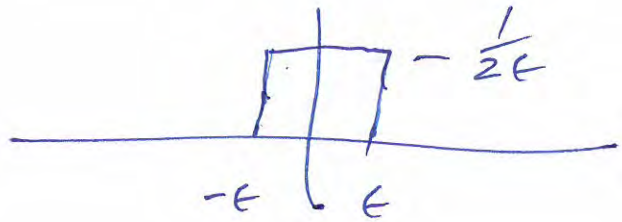
UNIT STEP AKA HEAVISIDE FCN





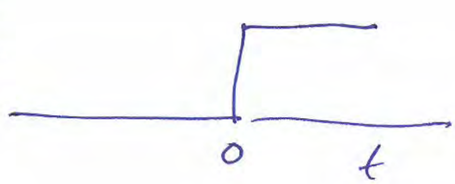
$\lim_{\epsilon \to 0}$

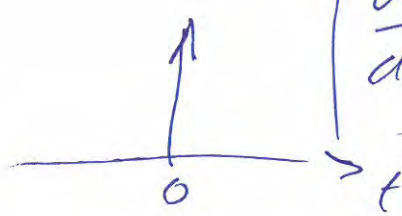
TAKE $\frac{d}{dt}$ OF THIS WAVEFORM



WHAT IS AREA? WHAT HAPPENS AS $\epsilon \to 0$?

$\lim\limits_{\epsilon \to 0}$  $= \delta(t)$ DIRAC δ-FUNCTION

$\frac{d}{dt}$ 

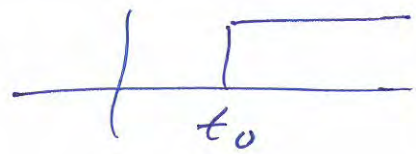
$=$ 

$$\boxed{\frac{du(t)}{dt} = \delta(t)}$$

ALSO $\int_{-\infty}^{t} \delta(t)\,dt = u(t)$

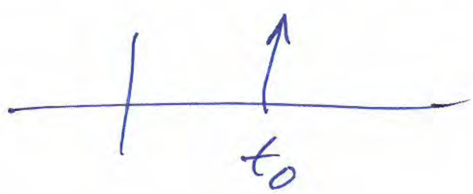
UNIT STEP TURNS ON AT t_0

$u(t-t_0)$



WHERE IS ARGUMENT ZERO?

DERIVATIVE IS



$\delta(t-t_0)$

$\mathcal{L}\{u(t)\} \stackrel{?}{=}$

$$\int_0^\infty u(t)e^{-st}\,dt = \int_{0^+}^\infty e^{-st}\,dt$$

$$= -\frac{1}{s}e^{-ts}\Big|_{0^+}^\infty = \frac{1}{s}$$

$$u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s}$$

$e^{-\alpha t} = f(t)$

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-\alpha t} e^{-st}\, dt = -\frac{1}{(\alpha+s)} e^{-(\alpha+s)t}\Big|_0^{\infty}$$

$$= \frac{1}{s+\alpha}$$

$$e^{-\alpha t} \xleftrightarrow{\mathcal{L}} \frac{1}{s+\alpha}$$

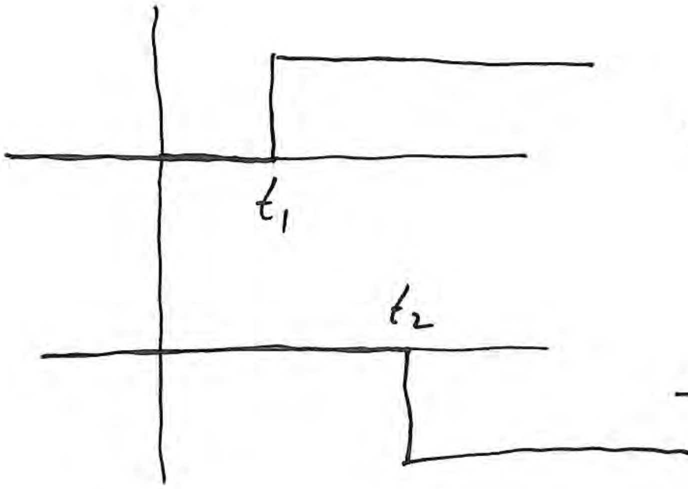
$$f(t) = \cos\omega t$$

$$\mathcal{L}\{\cos\omega t\} = \int_0^{\infty} \left(\frac{e^{j\omega t} + e^{-j\omega t}}{2} \right) e^{-st}\, dt$$

$$= \frac{1}{2}\int_0^{\infty} e^{-(s-j\omega)t}\, dt + \frac{1}{2}\int_0^{\infty} e^{-(s+j\omega)t}\, dt$$

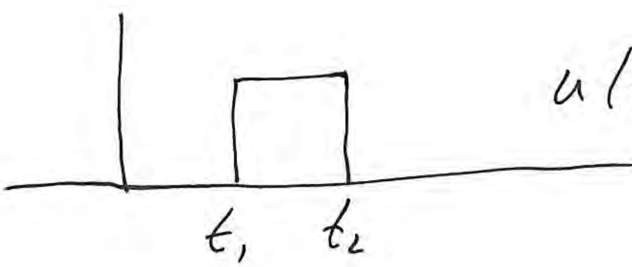
$$= \frac{1}{2}\,\frac{1}{s-j\omega} + \frac{1}{2}\,\frac{1}{s+j\omega} = \frac{s}{s^2+\omega^2}$$

CONSTRUCTION OF MORE COMPLICATED SIGNALS FROM UNIT STEP FCNS.

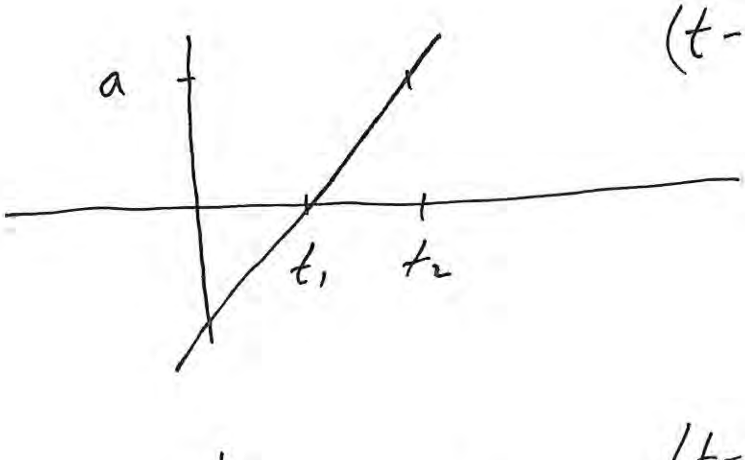


$u(t-t_1)$

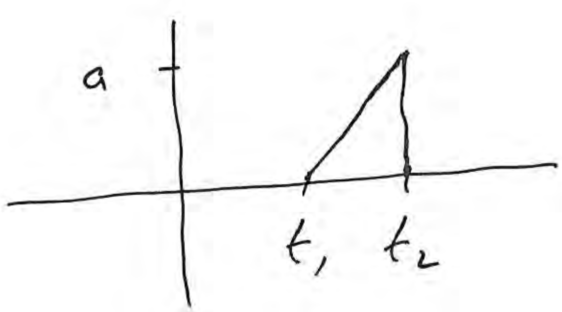
$-u(t-t_2)$



$u(t-t_1)-u(t-t_2)$

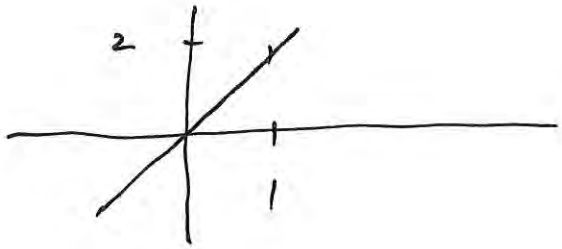


$(t-t_1)\dfrac{a}{(t_2-t_1)}$

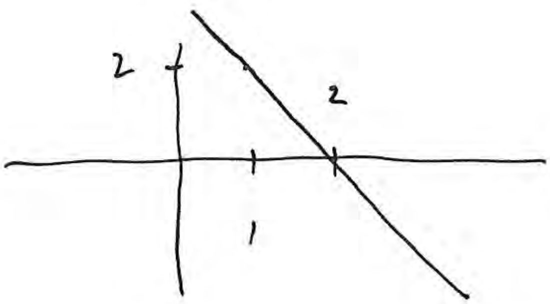


$(t-t_1)\dfrac{a}{(t_2-t_1)}\left[u(t-t_1)-u(t-t_2)\right]$

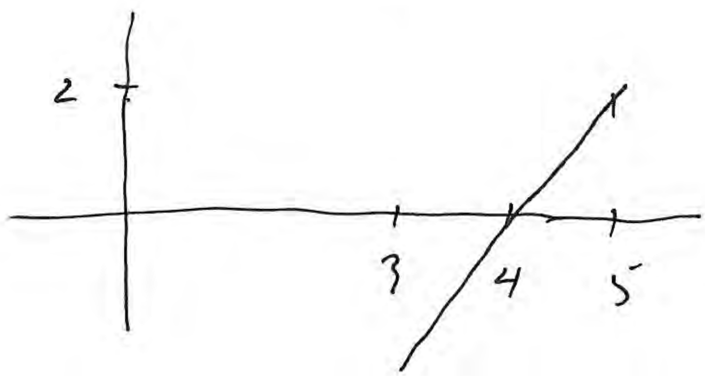
EXAMPLE 12.1



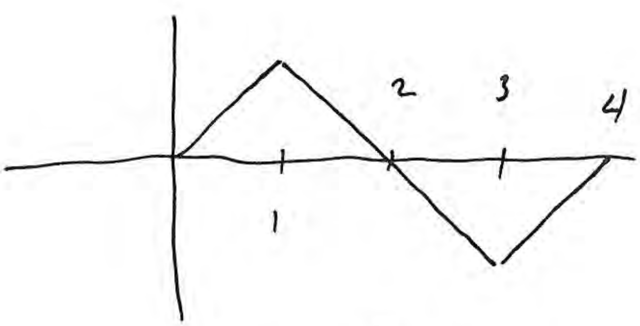
$f_1 = 2t$



$f_2 = -2(t-2)$



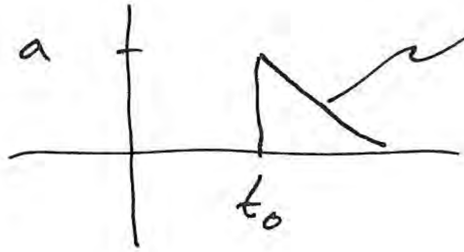
$f_3 = 2(t-4)$



$$2t\,[u(t)-u(t-1)]$$

$$+(-2t+4)\,[u(t-1)-u(t-3)]$$

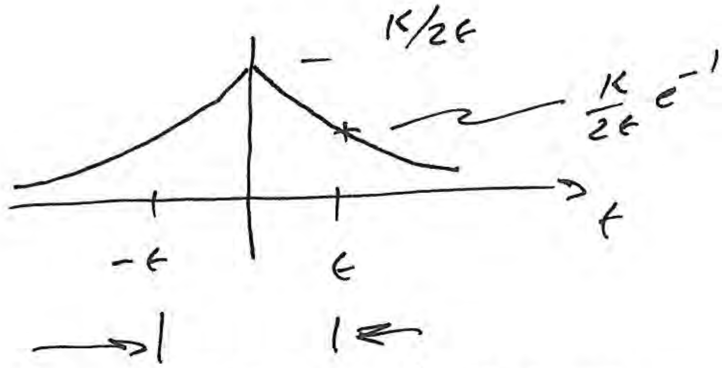
$$+(2t-8)\,[u(t-3)-u(t-4)]$$



$$f(t) = a e^{-\alpha(t-t_0)} u(t-t_0)$$

OTHER FUNCTIONS IN LIMIT $\longrightarrow \delta(t)$

$$f(t) = \frac{K}{2\epsilon} e^{-|t|/\epsilon}$$



WIDTH AT $1/e$ POINTS $= 2\epsilon$

AREA UNDER CURVE $= \int_{-\infty}^{\infty} f(t)\,dt$

$$= \frac{K}{2\epsilon} \int_{-\infty}^{\infty} e^{-|t|/\epsilon}\,dt$$

$$= \frac{K}{\epsilon} \int_{0}^{\infty} e^{-t/\epsilon}\,dt$$

$$= \frac{K}{\epsilon} \left(-\epsilon e^{-t/\epsilon} \Big|_0^{\infty} \right) = K$$

AS $\epsilon \to 0$, WIDTH $\to 0$

HEIGHT $\to \infty$

AREA REMAINS CONST

$$\lim_{\epsilon \to \cancel{\infty}\,\epsilon} \frac{K}{2\epsilon} e^{-|t|/\epsilon} = \delta(t) K$$

CLEARLY, $\int_{-\infty}^{\infty} K \delta(t) dt = K$

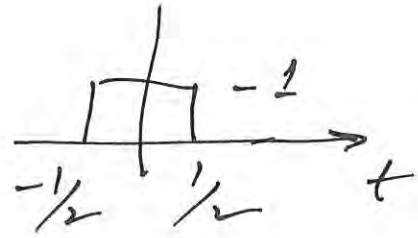
IMPORTANT "SIFTING" PROPERTY:

$$\int_{-\infty}^{\infty} f(t) \delta(t - t_0) dt = f(t_0)$$

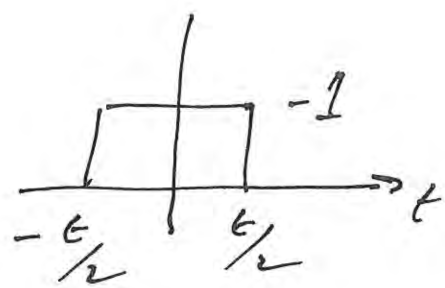
SIFTING PROPERTY OF δ-FUNCTION

$$\int_{-\infty}^{\infty} F(t)\,\delta(t-t_0)\,dt = f(t_0)$$

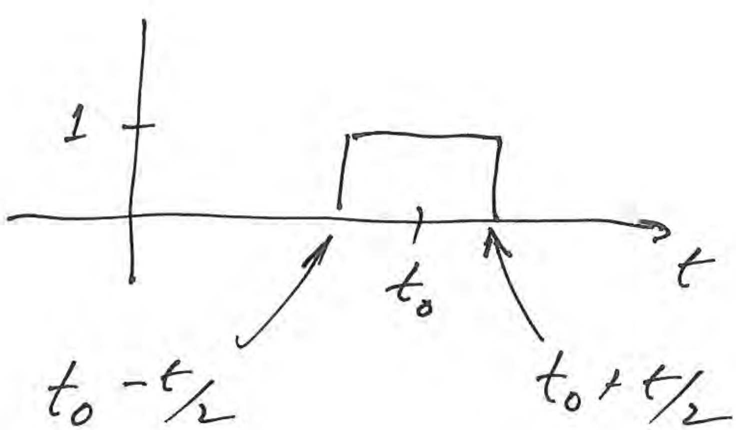
$$p(t) \equiv \begin{cases} 1 \;; & |t| \le 1/2 \\ 0 \;; & \text{ELSE} \end{cases}$$



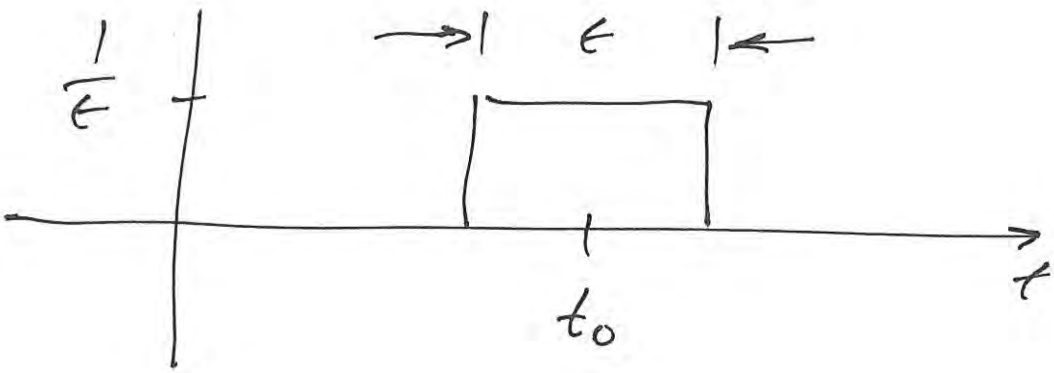
$$p(t/\epsilon) = \begin{cases} 1 \;; & |t/\epsilon| \le 1/2 \\ 0 \;; & \text{ELSE} \end{cases}$$



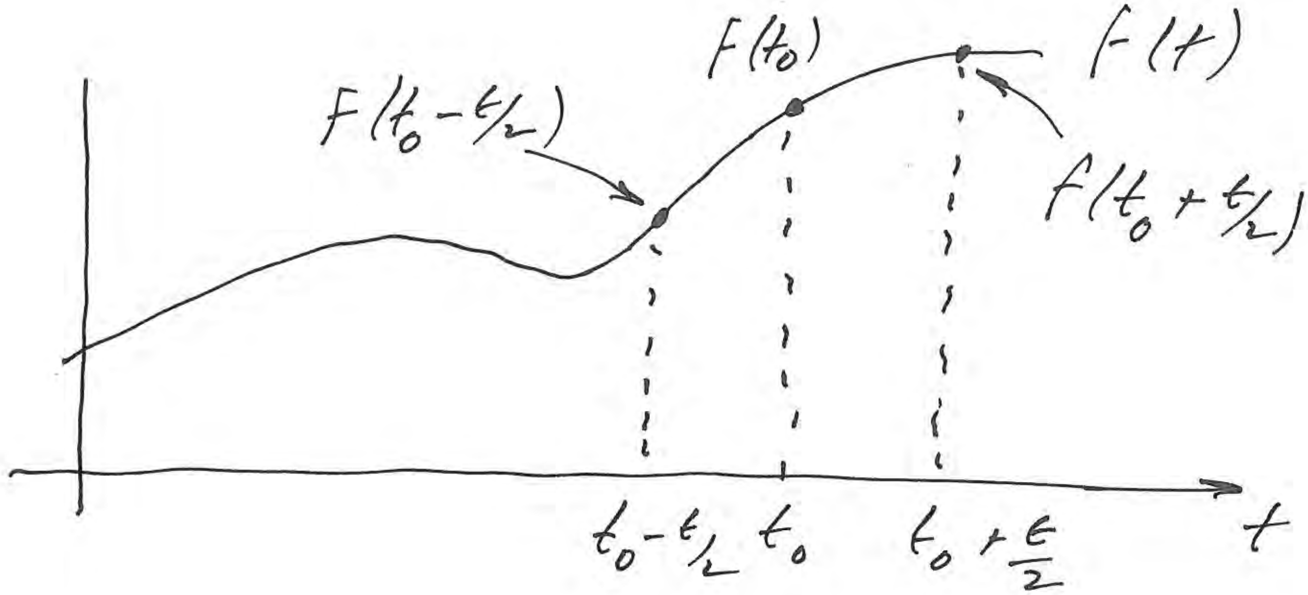
$$p\left(\frac{t-t_0}{\epsilon}\right) = \begin{cases} 1 \;; & \left|\frac{t-t_0}{\epsilon}\right| \le 1/2 \\ 0 \;; & \text{ELSE} \end{cases}$$



$$\frac{1}{\epsilon} p\left(\frac{t-t_0}{\epsilon}\right) = \begin{cases} \frac{1}{\epsilon} ; & \left|\frac{t-t_0}{\epsilon}\right| \le \frac{1}{2} \\ 0 ; & \text{ELSE} \end{cases}$$



CONSIDER $\displaystyle\int_{-\infty}^{\infty} f(t) \left[\frac{1}{\epsilon} p\left(\frac{t-t_0}{\epsilon}\right)\right] dt$



$$\int_{-\infty}^{\infty} F(t)\left[\frac{1}{\epsilon}\, p\left(\frac{t-t_0}{\epsilon}\right)\right] dt = \int_{t_0-\epsilon/2}^{t_0+\epsilon/2} F(t)\,\frac{1}{\epsilon}\, dt$$

$$\approx \frac{1}{\epsilon}\underbrace{\left[\frac{F(t_0+\epsilon/2)+F(t_0-\epsilon/2)}{2}\right]}_{\text{AVERAGE OF } F(t) \text{ WITHIN INTERVAL}} \epsilon$$

WIDTH OF INTERVAL

$$\int_{-\infty}^{\infty} f(t)\left[\frac{1}{\epsilon}\rho\left(\frac{t-t_0}{\epsilon}\right)\right]dt$$

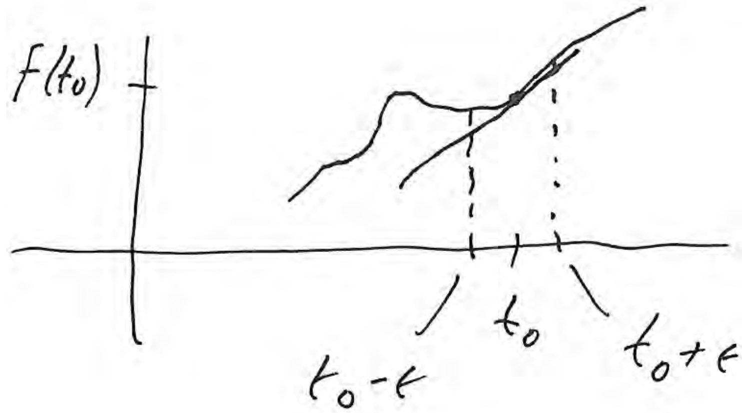
$$\approx \quad \frac{f(t_0+\epsilon/2)+f(t_0-\epsilon/2)}{2}$$

AS $\epsilon \to 0$, $\frac{1}{\epsilon}\rho\left(\frac{t-t_0}{\epsilon}\right) \longrightarrow \delta(t-t_0)$

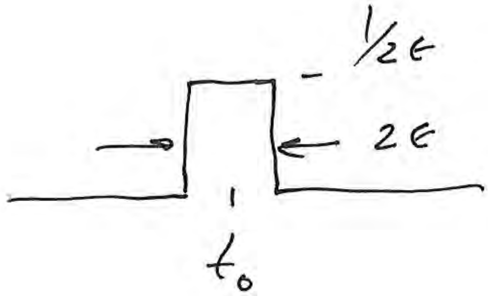
AND $\frac{f(t_0+\epsilon/2)+f(t_0-\epsilon/2)}{2} \longrightarrow f(t_0)$

$$\therefore \int_{-\infty}^{\infty} f(t)\,\delta(t-t_0)\,dt = f(t_0)$$

Q.E.D.



$$F(t_0) \approx \frac{f(t_0+\epsilon)+f(t_0-\epsilon)}{2}$$



VALUE OF INTEGRAL ≈

VALUE OF PEAK OF PULSE

$$\underbrace{2\epsilon}_{\text{WIDTH OF INTERVAL}} \underbrace{\left[\frac{f(t_0+\epsilon)+f(t_0-\epsilon)}{2}\right]}_{\text{VALUE OF FUNCTION}} \frac{1}{2\epsilon}$$

$$\frac{f(t_0+\epsilon)+f(t_0-\epsilon)}{2} \xrightarrow[\epsilon\to 0]{} f(t_0)$$

USE SIFTING PROPERTY TO FIND TRANSFORM OF $\delta(t)$

$$\mathcal{L}\{\delta(t)\} = \int_{0^-}^{\infty} e^{-st}\delta(t)\,dt$$

$$= e^{-st}\Big|_{0^-} = 1$$

$$\delta(t) \overset{\mathcal{L}}{\longleftrightarrow} 1$$

WHAT IS $\mathcal{L}\{u(t)\}$?

$$= \int_{0^-}^{\infty} u(t)e^{-st}\,dt = \int_{0^-}^{\infty} e^{-st}\,dt$$

$$= -\frac{1}{s}e^{-st}\Big|_{0^-}^{\infty} = \frac{1}{s}$$

$$u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s}$$

TRANSFORM OF DAMPED SINUSOID

$$\mathcal{L}\{e^{-\alpha t} \sin \omega t\}$$

$$e^{-\alpha t} \sin \omega t = e^{-\alpha t}\left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right)$$

$$= \frac{1}{2j}\left(e^{-t(\alpha - j\omega)} - e^{-t(\alpha + j\omega)}\right)$$

$$\mathcal{L}\{e^{-\alpha t} \sin \omega t\} = \int_{0^-}^{\infty} e^{-\alpha t} \sin \omega t \, e^{-st} \, dt$$

$$= \frac{1}{2j} \int_0^{\infty} e^{-t[s + (\alpha - j\omega)]} dt - \frac{1}{2j} \int_0^{\infty} e^{-t[s + (\alpha + j\omega)]} dt$$

$$= \frac{1}{2j}\left[\frac{1}{s + (\alpha - j\omega)} - \frac{1}{s + (\alpha + j\omega)}\right]$$

$$\vdots$$

$$\frac{\omega}{(s+\alpha)^2 + \omega^2}$$

$$e^{-\alpha t} \sin \omega t \xleftrightarrow{\mathcal{L}} \frac{\omega}{(s+\alpha)^2+\omega^2}$$

EASIER WAY? YES – STAY TUNED

OPERATIONAL TRANSFORMS

$$\mathcal{L}\{f(t)\} = F(s)$$

$$\mathcal{L}\{K f(t)\} = K F(s)$$

IMPLICATION:

IF $f_1 \longleftrightarrow F_1$ & $f_2 \longleftrightarrow F_2$

THEN $af_1 + bf_2 \longleftrightarrow aF_1 + bF_2$

(LINEARITY)

DIFFERENTIATION

$$\mathcal{L}\left\{\frac{d}{dt}f(t)\right\} = \int_{0^-}^{\infty} \frac{d}{dt}f(t)\, e^{-st}\, dt$$

BY PARTS (u dv SUBSTITUTION)

$$u = e^{-st} \qquad dv = \frac{d}{dt}f(t)\, dt$$

$$du = -s e^{-st} dt \qquad v = f(t)$$

$$= uv\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} v\, du$$

$$= e^{-st}f(t)\Big|_{0^-}^{\infty} + s\int_{0^-}^{\infty} f(t)e^{-st}dt$$

$$= -f(0^-) + s\,F(s)$$

IF $\quad f(t) \longleftrightarrow F(s)$

THEN $\quad f'(t) \longleftrightarrow sF(s) - f(0^-)$

— APPLICATION —

$$i(t) = C \frac{dv(t)}{dt}$$

$$I(s) = C\left[sV(s) - v(0^-)\right]$$

EASY TO SHOW THAT THIS GENERALIZES:

$$F'' \longleftrightarrow s^2 F(s) - s f(0^-) - \frac{d}{dt} f(0^-) \quad , \text{ETC.}$$

INTEGRATION

$$\mathcal{L}\left\{\int_{0^-}^{t} f(x)\,dx\right\} = \int_{0^-}^{\infty}\left[\int_{0^-}^{t} f(x)\,dx\right] e^{-st}\,dt$$

AGAIN BY PARTS

$$u = \int_{0^-}^{t} f(x)\,dx \qquad dv = e^{-st}\,dt$$

$$du = f(t)\,dt \qquad v = \frac{-e^{-st}}{s}$$

$$\mathcal{L}\{\cdots\} = \underbrace{\frac{-e^{-st}}{s}\int_{0^-}^{t} f(x)\,dx\Bigg|_{0^-}^{\infty}}_{\overset{\text{WHY ?}}{= 0}} + \underbrace{\frac{1}{s}\int_{0^-}^{\infty} f(t)e^{-st}\,dt}_{\frac{1}{s}F(s)}$$

$$\mathcal{L}\left\{\int_{0^-}^{t} F(x)\,dx\right\} = \frac{F(s)}{s}$$

SUMMARIZE IN WORDS, EFFECTS OF DIFFERENTIATION, INTEGRATION

TRANSLATION IN TIME DOMAIN

$$\mathcal{L}\{F(t-a)u(t-a)\} = e^{-as}F(s)$$

TRANSLATION MULTIPLICATION

$$\mathcal{L}\{F(t-a)u(t-a)\} = \int_{0^-}^{\infty} F(t-a)u(t-a)e^{-st}\,dt$$

$$= \int_{a}^{\infty} F(t-a)e^{-st}\,dt$$

cov: $x = t - a \quad dx = dt$

$$= \int_{0}^{\infty} F(x)e^{-s(x+a)}\,dx$$

— APPLICATION —

RECALL

$$i(t) = C \frac{dv(t)}{dt}$$

$$\vdots$$

$$I(s) = C[sV(s) - v(0^-)]$$

$$v(t) = \frac{1}{C}\int_0^t i(x)\,dx + v(0^-)$$

$$V(s) = \frac{1}{Cs} I(s) + \frac{v(0^-)}{s}$$

$$V(s) - \frac{v(0^-)}{s} = \frac{1}{Cs} I(s)$$

$$C[sV(s) - v(0^-)] = I(s)$$

$$\mathcal{L}\{f(t-a)\,u(t-a)\} = e^{-as}\underbrace{\int_0^{\infty} f(x)e^{-sx}\,dx}_{F(s)}$$

TRANSLATION IN FREQUENCY DOMAIN

$$\mathcal{L}\{e^{-at}f(t)\} \xleftrightarrow{\mathcal{L}} ?$$

$$\int_0^{\infty} e^{-at}f(t)e^{-st}\,dt = \underbrace{\int_0^{\infty} f(t)e^{-t(s+a)}\,dt}_{F(s+a)}$$

$$e^{-at}f(t) \longleftrightarrow F(s+a)$$

APPLICATION

$$\mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2+\omega^2} \quad \text{SINUSOID}$$

DAMPED SINUSOID

$$\mathcal{L}\{e^{-at} \sin \omega t\} = \frac{\omega}{(s+a)^2+\omega^2}$$

(WAY EASIER...)

SCALE CHANGES

$$\mathcal{L}\{F(at)\} = \int_{0^-}^{\infty} F(at) e^{-st}\, dt \qquad \begin{array}{c}\text{ASSUME} \\ a>0\end{array}$$

COV: $x = at \quad dx = a\,dt$

$$= \int_{0^-}^{\infty} F(x) e^{-sx/a} \frac{1}{a}\, dx$$

$$= \frac{1}{a} F(s/a)$$

IF $f(t) \longleftrightarrow F(s)$

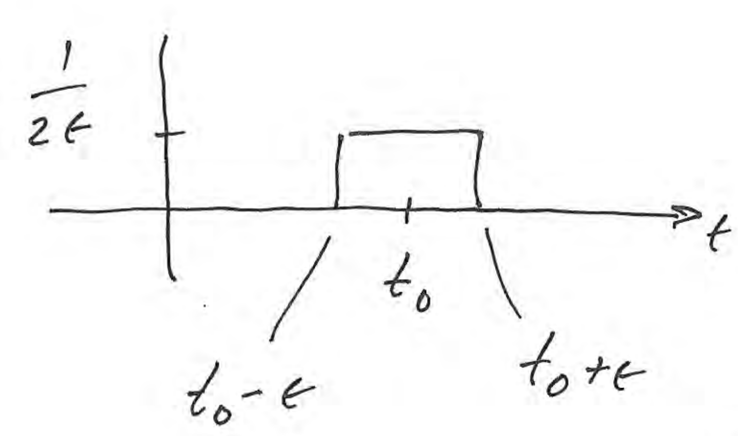
THEN $f(at) \longleftrightarrow \frac{1}{a} F\left(\frac{s}{a}\right)$ UNCERTAINTY PRINCIPLE

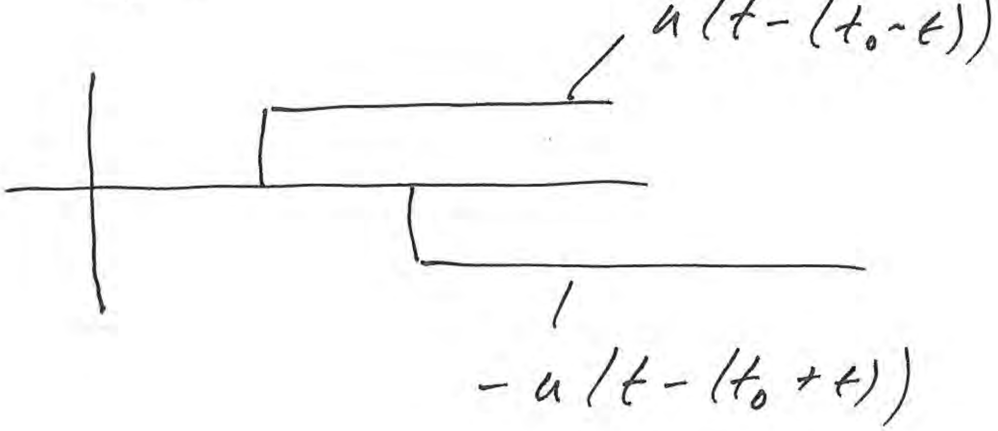
IF $a > 1$

THIS IS COMPRESSED AND THIS IS STRETCHED

THINK ABOUT, FOR EXAMPLE,

POINT AT WHICH $at = 1$ ...





$$u(t) \longleftrightarrow \frac{1}{s}$$

$$u(t-(t_0-\epsilon)) \longleftrightarrow \frac{e^{-s(t_0-\epsilon)}}{s}$$

$$\text{TRANSFORM} = \frac{1}{2\epsilon}\left[\frac{e^{-s(t_0-\epsilon)} - e^{-s(t_0+\epsilon)}}{s}\right]$$

$$= e^{-st_0}\underbrace{\left(\frac{e^{s\epsilon}-e^{-s\epsilon}}{2s\epsilon}\right)}_{\to\ 1 \text{ AS } \epsilon \to 0}$$

$\frac{\text{SINH}(s\epsilon)}{s\epsilon}$

(L'HOSPITAL'S RULE)

TRANSFORM PAIR

$$\delta(t-t_0) \overset{\mathcal{L}}{\longleftrightarrow} e^{-st_0}$$

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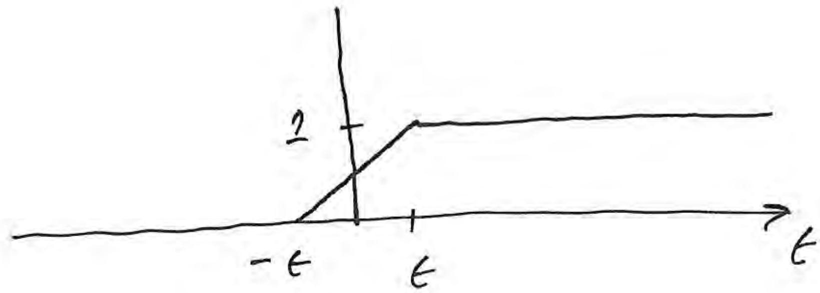
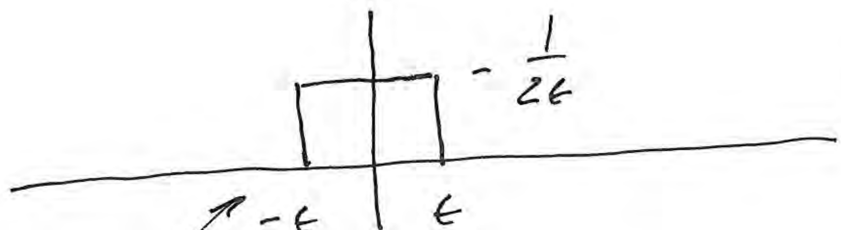
$$\int_{-\infty}^{\infty} F(t)\, u(t-t_0)\, u(t_1-t)\, dt = ?$$



FUNCTION



DERIVATIVE

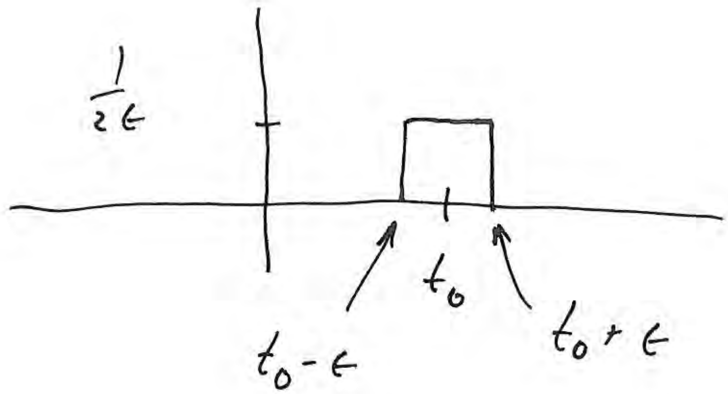
AREA = 1

CONSIDER LIMIT AS $\epsilon \to 0$

$$\frac{d}{dt} u(t) = \delta(t)$$

SIZE OF DISCONTINUITY IS 1

AREA OF δ-FUNCTION IS 1

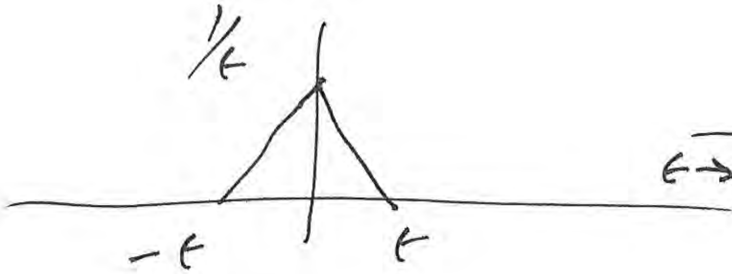


$$= \frac{1}{2\epsilon}\left[u(t-(t_0-\epsilon)) - u(t-(t_0+\epsilon))\right]$$

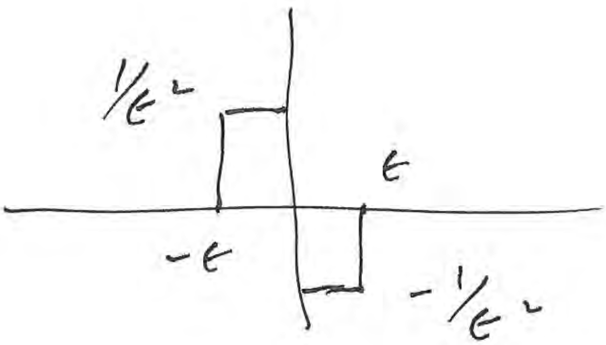
$$\left(\begin{array}{l} u(t) \longleftrightarrow \frac{1}{s} \\ F(t-a) \longleftrightarrow e^{-as} F(s) \end{array} \right)$$

$$= \frac{1}{2\epsilon}\left[\frac{e^{-(t_0-\epsilon)s}}{s} - \frac{e^{-(t_0+\epsilon)s}}{s} \right]$$

$$= e^{-t_0 s} \frac{1}{\epsilon s}\left(\frac{e^{\epsilon s} - e^{-\epsilon s}}{2} \right) = e^{-t_0 s} \frac{\sinh(\epsilon s)}{\epsilon s}$$



$$\xrightarrow[\epsilon \to 0]{} \delta(t)$$



$$\xrightarrow[\epsilon \to 0]{} \delta'(t)$$

KNOW $\delta(t) \longleftrightarrow 1$

DERIVATIVE THEOREM SAYS

$$\delta'(t) \longleftrightarrow s$$

PROPERTY OF $\delta'(t)$

CONSIDER $\displaystyle\int_{-\infty}^{\infty} f(t)\,\delta'(t)\,dt$

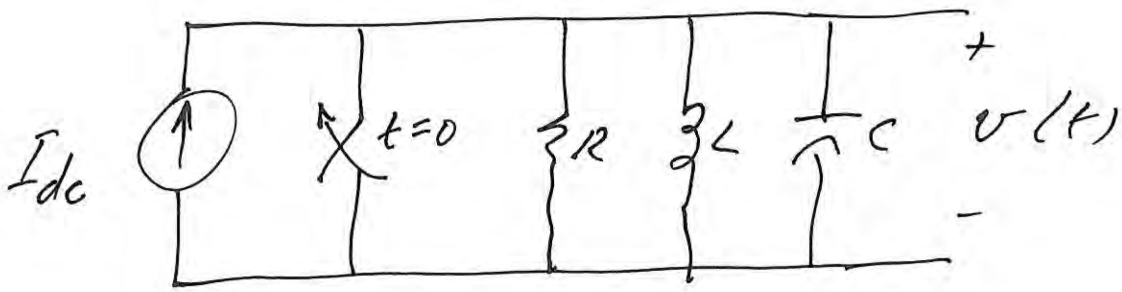
LET $u = f(t)$ $\quad dv = \delta'(t)\,dt$

$du = f'(t)\,dt \quad v = \delta(t)$

$$= \cancelto{0}{f(t)\delta(t)}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(t)\,\delta(t)\,dt$$

$$= -f'(0)$$

APPLICATIONS



ASSUME NO STORED ENERGY

NODAL ANALYSIS:

$$\frac{v(t)}{R} + \frac{1}{L}\int_0^t v(x)\,dx + C\frac{dv(t)}{dt} = I_{dc}\,u(t)$$

taking $\mathcal{L}\{\cdots\}$ yields

$$\frac{V(s)}{R} + \frac{V(s)}{Ls} + CsV(s) = I_{dc}\,\frac{1}{s}$$

$$V(s) = \frac{I_{dc}(1/s)}{\left(\frac{1}{R} + \frac{1}{Ls} + Cs\right)}$$

$$= \frac{I_{dc}/C}{s^2 + (1/RC)s + (1/LC)}$$

FORMALLY, WE NEED

$$v(t) = \mathcal{L}^{-1}\left\{ \frac{I_{dc}/C}{s^2 + (1/RC)s + (1/LC)} \right\}$$

PROBLEM: THIS DOESN'T RESEMBLE ANYTHING IN OUR TABLE OF TRANSFORMS.

CAN WE MAKE IT LOOK LIKE?

AS SUGGESTED BY THE PREVIOUS EXAMPLE, THE DENOMINATOR OF THE FUNCTION WE NEED TO TAKE THE INVERSE TRANSFORM OF, IS A POLYNOMIAL IN S.

MORE GENERALLY, THE FUNCTION WE NEED TO INVERSE TRANSFORM IS A QUOTIENT OF POLYNOMIALS,

$$F(s) = \frac{N(s)}{D(s)} = \frac{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0}{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}$$

n & m ARE THE (INTEGER) "DEGREES" OF THE POLYNOMIALS

THERE ARE TWO GENERAL SOLUTION APPROACHES DEPENDING ON WHETHER $m \leq n$ OR NOT.

IF $m \leq n$ $F(s)$ IS A "PROPER" RATIONAL FUNCTION

IF $m > n$ $F(s)$ IS AN "IMPROPER" RATIONAL FCN

WE'LL EXPLORE THE FORMER CASE FIRST

PROPER RATIONAL FUNCTIONS

THE ANALYSIS APPROACH FOR THIS CASE, "PARTIAL FRACTION EXPANSION," HINGES ON FINDING THE ROOTS OF THE POLYNOMIAL $\underline{\underline{D(S)}}$.

RECALL THAT IN OUR ANALYSIS OF TRANSIENT EFFECTS, WE DETERMINED THE ROOTS OF THE CHARACTERISTIC EQ, AND THAT THE VALUES OF THESE ROOTS DICTATED THE CIRCUIT BEHAVIOR.

MUCH IS THE SAME HERE.

EXAMPLE

$$\text{SAY} \quad F(s) = \frac{s+6}{s(s+3)(s+1)^2}$$

NOTE THAT $D(s)$ IS IN FACTORED FORM. IN EFFECT, AT THIS POINT WE HAVE IDENTIFIED THE ROOTS:

$$s = 0$$
$$s = -3$$
$$s = -1 \quad \text{(DOUBLE ROOT)}$$

HOW WE ARRIVE AT THIS FACTORED FORM FROM THE GENERAL POLYNOMIAL WILL BE DISCUSSED. NOTE THAT WE KNOW HOW TO DO THIS IF THE DEGREE OF THE POLYNOMIAL $D(s)$ IS <u>TWO</u>...

CLAIM: ONCE FACTORED, WE CAN WRITE

$$\frac{s+6}{s(s+3)(s+1)^2} = \frac{K_1}{s} + \frac{K_2}{s+3} + \underbrace{\frac{K_3}{(s+1)^2} + \frac{K_4}{(s+1)}}$$

NOTE THE PATTERN

THESE TWO BECAUSE OF THE REPEATED ROOT

ASSUMING THAT WE CAN DO THIS, OBSERVE THE INDIVIDUAL TERMS

$$\frac{K_1}{s} \longleftrightarrow K_1 u(t)$$

$$\frac{K_2}{s+3} \longleftrightarrow K_2 e^{-3t} u(t)$$

$$\frac{K_3}{(s+1)^2} = -K_3 \frac{d}{dt} \frac{1}{s+1}$$

$$t f(t) \longleftrightarrow -\frac{d}{ds} F(s)$$

$$\therefore \quad \frac{K_3}{(s+1)^2} \longleftrightarrow K_3 t e^{-t} u(t)$$

$$\frac{K_4}{s+1} \longleftrightarrow K_4 e^{-t} u(t)$$

$$\mathcal{L}^{-1}\left\{ \frac{s+6}{s(s+3)(s+1)^2} \right\} =$$

$$\left(K_1 + K_2 e^{-3t} + K_3 t e^{-t} + K_4 e^{-t}\right) u(t)$$

NOTE: THIS WORKS <u>ONLY</u> IF $F(s)$ IS A "PROPER" RATIONAL FUNCTION

$\longrightarrow$ DEGREE OF $D(s) \geq$ DEGREE OF $N(s)$

IN THIS EXAMPLE DEGREE OF $N(s) = 1$

WHAT IS DEGREE OF $D(s)$?

SO, HOW DO WE DETERMINE THE K_j'S ?

EXAMPLE (DISTINCT REAL ROOTS)

$$F(s) = \frac{96(s+5)(s+12)}{s(s+8)(s+6)} = \frac{K_1}{s} + \frac{K_2}{s+8} + \frac{K_3}{s+6}$$

STEP 1 : MULTIPLY BOTH SIDES BY $\underline{\underline{s}}$ AND SET $s = \underline{\underline{0}}$ ON BOTH SIDES

$$\frac{96(s+5)(s+12)}{(s+8)(s+6)} = K_1 + \frac{K_2 s}{s+8} + \frac{K_3 s}{s+6}$$

$$120 = \frac{96(5)(12)}{(8)(6)} = K_1 + 0 + 0$$

STEP 2 : MULTIPLY BOTH SIDES BY $\underline{\underline{s+8}}$ AND SET $\underline{\underline{s=-8}}$ ON BOTH SIDES

$$\frac{96(s+5)(s+12)}{s\ (s+6)} = \frac{K_1(s+8)}{s} + K_2 + \frac{K_3(s+8)}{s+6}$$

$$-72 = \frac{96(-8+5)(-8+12)}{(-8)(-8+6)} = 0 + K_2 + 0$$

STEP 3: MULTIPLY BOTH SIDES BY $\underline{\underline{s+6}}$

AND SET $\underline{\underline{s=-6}}$ ON BOTH SIDES

$\vdots$

$$48 = \frac{96(-6+5)(-6+12)}{(-6)(-6+8)} = K_3$$

$$\frac{96\ (s+5)(s+12)}{s\ (s+8)\ (s+6)} = \frac{120}{s} - \frac{72}{s+8} + \frac{48}{s+6}$$

NOTE: WANT TO CHECK RESULT

HOW?

REALIZE THIS IS AN IDENTITY

WHAT DOES THIS MEAN?

FINALLY;

$$\mathcal{L}^{-1}\left\{\frac{96(s+5)(s+12)}{s(s+8)(s+6)}\right\} = \mathcal{L}^{-1}\left\{\frac{120}{s} - \frac{72}{s+8} + \frac{48}{s+6}\right\}$$

$$= 120u(t) - 72e^{-8t}u(t) + 48e^{-6t}u(t)$$

ASSESSMENT PROBLEMS 12.3, & 12.4

ASSESSMENT

12.3

$$F(s) = \frac{6s^2+26s+26}{(s+1)(s+2)(s+3)}$$

PROPER? YES

$$\frac{6s^2+26s+26}{(s+1)(s+2)(s+3)} = \frac{K_1}{s+1} + \frac{K_2}{s+2} + \frac{K_3}{s+3}$$

$$\left.\frac{(6s^2+26s+26)}{(s+2)(s+3)}\right|_{s=-1} = K_1 = 3$$

$$\left.\frac{(6s^2+26s+26)}{(s+1)(s+3)}\right|_{s=-2} = K_2 = 2$$

$$\left.\frac{(6s^2+26s+26)}{(s+1)(s+2)}\right|_{s=-3} = K_3 = 1$$

$$f(t) = (3e^{-t} + 2e^{-2t} + e^{-3t})u(t)$$

ASSESSMENT
12.4

12

$$F(s) = \frac{7s^2+63s+134}{(s+3)(s+4)(s+5)}$$ PROPER ?

$$\frac{7s^2+63s+134}{(s+3)(s+4)(s+5)} = \frac{K_1}{s+3} + \frac{K_2}{s+4} + \frac{K_3}{s+5}$$

$$\left.\frac{(7s^2+63s+134)}{(s+4)(s+5)}\right|_{s=-3} = K_1 = 4$$

$$\left.\frac{(7s^2+63s+134)}{(s+3)(s+5)}\right|_{s=-4} = K_2 = 6$$

$$\left.\frac{(7s^2+63s+134)}{(s+3)(s+4)}\right|_{s=-5} = K_3 = -3$$

$$f(t) = (4e^{-3t} + 6e^{-4t} - 3e^{-5t})u(t)$$

CASE: DISTINCT COMPLEX ROOTS

GUESS WHAT RESULTS WILL BE -

DAMPED, OSCILLATORY, COMBINATION?

EXAMPLE

$$F(s) = \frac{100(s+3)}{(s+6)\underbrace{(s^2+6s+25)}_{\text{FACTOR THIS}}} \qquad \text{PROPER?}$$

$$s = \frac{-6 \pm \sqrt{36-4(25)}}{2} = -3 \pm \sqrt{-16}$$

$$s = -3 \pm j4$$

$$F(s) = \frac{100(s+3)}{(s+6)[s+(3-j4)][s+(3+j4)]}$$

NOTE ROOTS DISTINCT - CAN USE PREVIOUS METHOD

$$F(s) = \frac{K_1}{s+6} + \frac{K_2}{s+3-j4} + \frac{K_3}{s+3+j4}$$

$$\left.\frac{100(s+3)}{(s+6)(s+3+j4)}\right|_{s=-3+j4} = K_2$$

$$\frac{100(-3+j4+3)}{(-3+j4+6)(-3+j4+3+j4)} = \frac{100(j4)}{(3+j4)(j8)}$$

$$= \frac{50}{3+j4}\frac{(3-j4)}{(3-j4)} = 2(3-j4)$$

$$K_2 = 6 - j8$$

$$K_2^* = 6 + j8$$

$$F(s) = \frac{-12}{s+6} + \underbrace{\frac{6-j8}{s+3-j4} + \frac{6+j8}{s+3+j4}}$$

RECALL THAT FOR X COMPLEX,

$$Re\{X\} = \frac{X+X^*}{2}$$

EACH TERM OF FORM

$$\frac{1}{s+a} \longleftrightarrow e^{-at}$$

COMPLEX TERMS ONLY

$$\frac{K}{s+3-j4} + \frac{K^*}{s+3+j4} \longleftrightarrow$$

$$\left[K e^{-(3-j4)t} + K^* e^{-(3+j4)t}\right] u(t)$$

$$K = 6-j8 = |K| e^{j\theta}$$

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COMPLEX TERMS ONLY

$$= \left[|K| e^{j\theta} e^{-(3-j4)t} + |K| e^{-j\theta} e^{-(3+j4)t} \right] u(t)$$

$$= |K| e^{-3t} \left(e^{j(4t+\theta)} + e^{-j(4t+\theta)} \right) u(t)$$

$$= 2|K| e^{-3t} \cos(4t+\theta)\, u(t)$$

$$\frac{e^{ja} + e^{-ja}}{2} = \cos(a)$$

REPEATED REAL ROOTS

EXAMPLE

$$\frac{100(s+25)}{s(s+5)^3} = \frac{K_1}{s} + \frac{K_2}{(s+5)^3} + \frac{K_3}{(s+5)^2} + \frac{K_4}{s+5}$$

PROPER?

NOTE PATTERN

CLAIM: THIS IS AN IDENTITY

WHAT IS THE IMPLICATION?

AS BEFORE FOR K_1:

$$\left.\frac{100(s+25)}{(s+5)^3}\right|_{s=0} = K_1 + \left.\frac{K_2 s}{(s+5)^3}\right|_{s=0} + \left.\frac{K_3 s}{(s+5)^2}\right|_{s=0} + \left.\frac{K_4 s}{(s+5)}\right|_{s=0}$$

$$20 = K_1$$

PREVIOUS PROCEDURE WORK FOR K_2 ?

$$\frac{100(s+25)}{s} = \frac{K_1(s+5)^3}{s} + K_2 + K_3(s+5) + K_4(s+5)^2$$

SET $s = -5$ BOTH SIDES

$$\frac{100(20)}{-5} = -400 = K_2$$

OK

HOW ABOUT K_3 ?

$$\frac{100(s+25)}{s(s+5)} = \frac{K_1(s+5)^2}{s} + \frac{K_2}{(s+5)} + K_3 + K_4(s+5)^2$$

PROBLEM: CAN'T MAKE THESE TERMS DISAPPEAR

NEED DIFFERENT APPROACH:
RETURN TO ☆

$$\frac{100(s+25)}{s} = \frac{K_1(s+5)^3}{s} + K_2 + K_3(s+5) + K_4(s+5)^2$$

THIS IS AN IDENTITY.

WHAT DOES THIS MEAN?

TRY TAKING $\frac{d}{ds}$ OF BOTH SIDES

$$LHS = \frac{100[s-(s+25)]}{s^2} = \frac{-2{,}500}{s^2}$$

$$RHS = K_1 \frac{d}{ds}\frac{(s+5)^3}{s} + \frac{d}{ds}K_2 + K_3\frac{d}{ds}(s+5)$$

$$+ K_4 \frac{d}{ds}(s+5)^2$$

$$= K_1 \frac{[3(s+5)^2 - (s+5)^3]}{s^2} + K_2(0) + K_3$$

$$+ K_4 2(s+5)$$

$$\bigstar\bigstar \quad \frac{-2,500}{s^2} = K_1 \frac{\left[3(s+5)^2 - (s+5)^3\right]}{s^2} + K_3 + K_4\, 2(s+5)$$

<u>NOW</u> SET $s = -5$

$$-\frac{2,500}{25} = K_1(0) + K_3 + K_4(0)$$

$$K_3 = -100$$

PROCEDURE SUGGESTS HOW TO GET K_4:

DIFFERENTIATE $\bigstar\bigstar$ WRT s.

$$\frac{5,000}{s^3} = \frac{d}{ds}\left\{\frac{\left[3(s+5)^2 - (s+5)^3\right]}{s^2}\right\} + \cancel{\frac{d}{ds}K_3}^{\,0} + 2K_4$$

SET $s = -5$ BOTH SIDES

$$\frac{5,000}{(-5)^3} = 2K_4 \rightarrow K_4 = -20$$

$$\frac{100(s+25)}{s(s+5)^3} = \frac{20}{s} - \frac{400}{(s+5)^3} - \frac{100}{(s+5)^2} - \frac{20}{s+5}$$

VALID EXPANSION ?

REALLY EASY CHECK IS FOR $s = -25$

$LHS = 0$

$$RHS = \frac{20}{-25} - \frac{400}{(-20)^3} - \frac{100}{(-20)^2} + \frac{20}{20}$$

$$= -\frac{4}{5} + \frac{1}{20} - \frac{1}{4} + 1$$

$$= \frac{-16 - 5 + 1 + 20}{20} = 0 \quad \checkmark$$

$$f(t) = \left(20 - 400t^2 e^{-5t} - 100te^{-5t} - 20e^{-5t}\right)u(t)$$

USED $\quad tf(t) \longleftrightarrow -\frac{dF(s)}{ds}$

REFER TO TABLE 12.3 FOR

SUMMARY OF RESULTS FOR

DISTINCT REAL

REPEATED REAL

DISTINCT COMPLEX

REPEATED COMPLEX

ASSESSMENT 12.7

$$F(s) = \frac{40}{(s^2+4s+5)^2}$$

ROOTS ARE $s_i = \frac{-4 \pm \sqrt{16-20}}{2}$

$$= -2 \pm j1$$

$$F(s) = \frac{40}{(s+2-j1)^2(s+2+j1)^2}$$

REPEATED COMPLEX

FROM TABLE 12.3

$$\alpha = 2, \ \beta = 1$$

BUT WHAT IS K ?

$$\frac{40}{(s+2-j)^2(s+2+j)^2} = \frac{K_1}{(s+2-j)^2} + \frac{K_1^*}{(s+2+j)^2}$$

$$+ \frac{K_2}{s+2-j} + \frac{K_2^*}{s+2+j}$$

FOR K_1 MULTIPLY BY $(s+2-j)^2$

$$\frac{40}{(s+2+j)^2} = K_1 + \frac{K_1^*(s+2-j)^2}{(s+2+j)^2}$$

$$+ K_2(s+2-j) + \frac{K_2^*(s+2-j)^2}{s+2+j}$$

SET $s = -2+j$

$$\frac{40}{(-2+j+2+j)^2} = K_1 = \frac{40}{(2j)^2} = -10$$

FOR K_2 DIFFERENTIATE * WRT S

$$LHS = \frac{40\left[-2(s+2+j)\right]}{(s+2+j)^4} = \frac{-80}{(s+2+j)^3}$$

$$LHS\Big|_{s=-2+j} = \frac{-80}{(-2+j+2+j)^3} = \frac{-80}{(j2)^3}$$

$$= \frac{-80}{-j8} = \frac{80}{j8} = \frac{10}{j} = -j10$$

EASY TO SEE THAT ONLY TERM ON RHS

IS K_2

$$F(s) = \frac{-10}{(s+2-j)^2} - \frac{10}{(s+2+j)^2}$$

$$-\frac{j10}{s+2-j} + \frac{j10}{s+2+j}$$

$$\frac{-10}{(s+2-j)^2} - \frac{10}{(s+2+j)^2} \longleftrightarrow -20te^{-2t}\cos(t)u(t)$$

$$\frac{-j10}{s+2-j} + \frac{j10}{s+2+j} \longleftrightarrow 20e^{-2t}\cos(t-90^\circ)u(t)$$

$$f(t) = (-20te^{-2t}\cos t + 20e^{-2t}\sin t)u(t)$$

IMPROPER RATIONAL FUNCTIONS

EXAMPLE PG 453

$$F(s) = \frac{s^4 + 13s^3 + 66s^2 + 200s + 300}{s^2 + 9s + 20}$$

DEGREE $\underline{\underline{4}}$

DEGREE $\underline{\underline{2}}$

DIVIDE D(S) INTO N(S):

$$\begin{array}{r|l}
 & s^2 + 4s + 10 \\
\hline
s^2+9s+20 & s^4 + 13s^3 + 66s^2 + 200s + 300 \\
 & s^4 + 9s^3 + 20s^2 \\
\cline{2-2}
 & \quad 4s^3 + 46s^2 + 200s \\
 & \quad 4s^3 + 36s^2 + 80s \\
\cline{2-2}
 & \qquad 10s^2 + 120s + 300 \\
 & \qquad 10s^2 + 90s + 200 \\
\cline{2-2}
 & \qquad\qquad 30s + 100
\end{array}$$

REMAINDER

$$F(s) = s^2 + 4s + 10 + \frac{30s + 100}{s^2 + 9s + 20}$$

PROPER RATIONAL FUNCTION

$$\frac{30s+100}{(s+4)(s+5)} = \frac{K_1}{s+4} + \frac{K_2}{s+5}$$

$\vdots$

$$F(s) = s^2 + 4s + 10 - \frac{20}{s+4} + \frac{50}{s+5}$$

$\updownarrow$

$$f(t) = \delta''(t) + 4\delta'(t) + 10\delta(t) - \left(20e^{-4t} - 50e^{-5t}\right)u(t)$$

$$\frac{30s+100}{(s+4)(s+5)} = \frac{K_1}{s+4} + \frac{K_2}{s+5}$$

$$30s+100 = K_1(s+5) + K_2(s+4)$$

$$s(K_1+K_2-30) + (5K_1+4K_2-100) = 0$$

$$K_1+K_2 = 30 \longrightarrow K_1 = 30-K_2$$

$$5K_1+4K_2 = 100$$

$$5(30-K_2)+4K_2 = 100$$

$$150-5K_2+4K_2 = 100 \longrightarrow \underline{\underline{K_2 = 50}}$$

$$\underline{\underline{K_1 = 30-K_2 = -20}}$$

POLES & ZEROS OF $F(s)$

WE WRITE $F(s)$ IN FACTORED FORM:

$$F(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_n)}{(s+p_1)(s+p_2)\cdots(s+p_m)}$$

ROOTS OF THE DENOMINATOR,

$$s = -p_1, -p_2, \cdots, -p_m$$

ARE CALLED POLES OF $F(s)$

BECAUSE $F(s)$ BLOWS UP AT THESE POINTS

ROOTS OF THE NUMERATOR,

$$s = -z_1, -z_2, \cdots, -z_n$$

ARE CALLED ZEROS OF $F(s)$

BECAUSE . . .

EXAMPLE

$$F(s) = \frac{8s^2 + 120s + 400}{2s^4 + 20s^3 + 70s^2 + 100s + 48}$$

$$= \frac{8(s^2 + 15s + 50)}{2(s^4 + 10s^3 + 35s^2 + 50s + 24)}$$

$$= \frac{4(s^2 + 15s + 50)}{s^4 + 10s^3 + 35s^2 + 50s + 24}$$

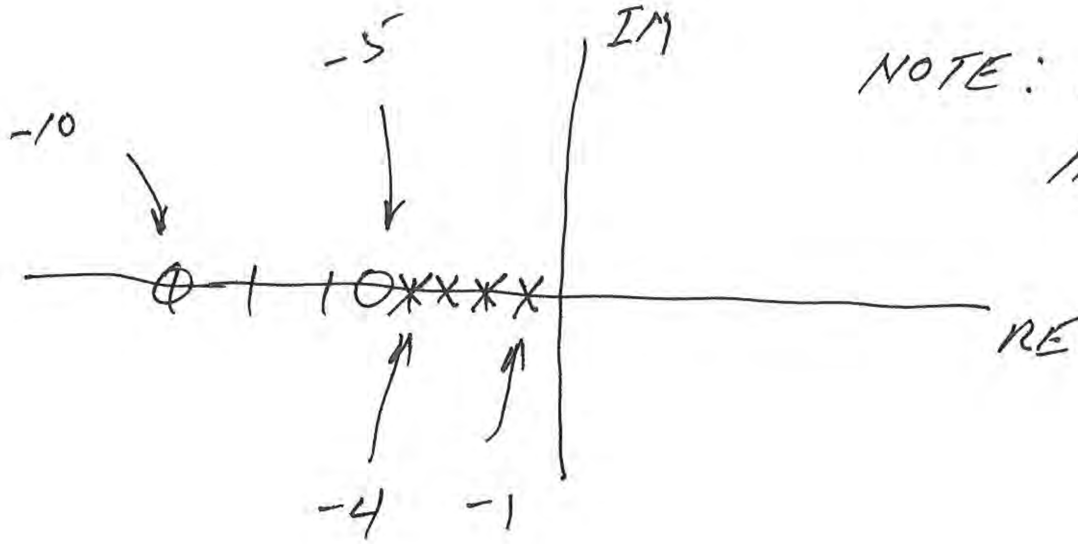
$$= \frac{4(s+5)(s+10)}{(s+1)(s+2)(s+3)(s+4)}$$

ZEROS: $-5, -10$

POLES: $-1, -2, -3, -4$

ILLUSTRATE LOCATIONS

IN COMPLEX PLANE



NOTE: ALL POLES IN L.H.P.

NOTE: TIME DOMAIN SIGNAL OF FORM

$$f(t) = \left(C_1 e^{-t} + C_2 e^{-2t} + C_3 e^{-3t} + C_4 e^{-4t}\right) u(t)$$

ALL TERMS DAMPED

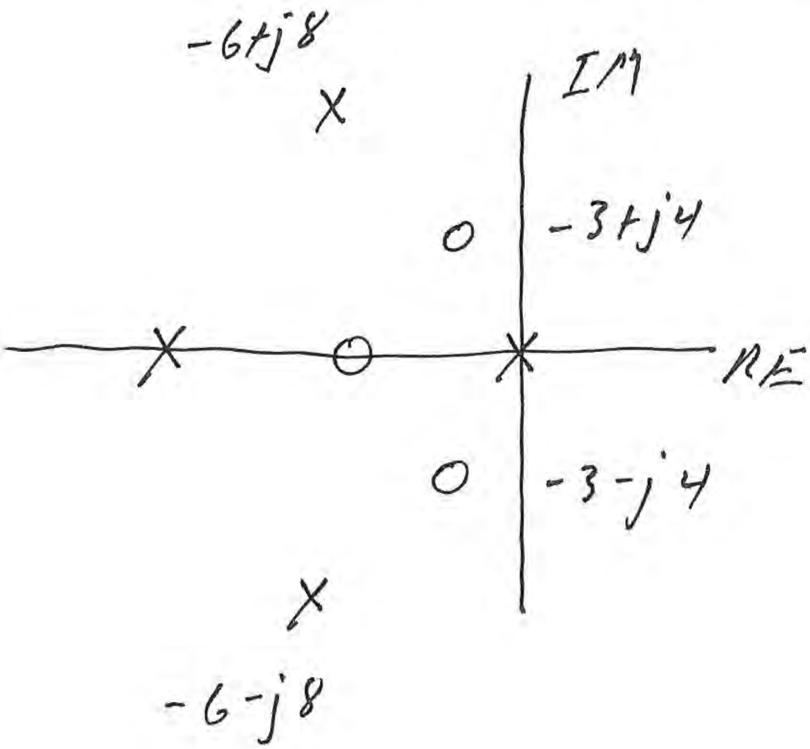
WHAT WOULD HAPPEN IF WE HAD POLES IN R.H.P. ?

EXAMPLE

$$F(s) = \frac{10(s+5)(s+3-j4)(s+3+j4)}{s(s+10)(s+6-j8)(s+6+j8)}$$

OBSERVATION: COMPLEX POLES, & COMPLEX ZEROS

COME IN COMPLEX CONJUGATE PAIRS



FOUR POLES

THREE ZEROS ?

ALL POLES

IN L.H.P.

IMPLICATIONS ?

HOW MANY POLES ?

HOW MANY ZEROS ? (TRICK QUESTION)

INITIAL & FINAL VALUES

F(S) DICTATES BEHAVIOR OF f(t) FOR ALL t

CAN WE INFER INITIAL ($f(0)$) AND FINAL ($f(\infty)$) VALUES FROM F(S)? YES (BUT HOW?)

CLAIM: $\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)$

INITIAL VALUE THEOREM

CLAIM: $\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$

FINAL VALUE THEOREM

PROOF OF IVT

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = \underbrace{sF(s) - f(0^-)}_{\text{ALREADY PROVED THIS}} = \underbrace{\int_{0^-}^{\infty} \frac{df}{dt} e^{-st}\, dt}_{\text{DEFINITION OF LAPLACE TRANSF.}}$$

TAKE $\lim\limits_{s\to\infty}$ OF BOTH SIDES

$$LHS = \lim_{s\to\infty} \left[sF(s) - f(0^-) \right]$$

$$RHS = \lim_{s\to\infty} \int_{0^-}^{\infty} \frac{df}{dt} e^{-st}\, dt$$

$$= \lim_{s\to\infty} \left\{ \underbrace{\int_{0^-}^{0^+} \frac{df}{dt} e^{-st}\, dt}_{f(0^+) - f(0^-)} + \int_{0^+}^{\infty} \frac{df}{dt} e^{-st}\, dt \right\}$$

WHY ?

$$RHS = f(0^+) - f(0^-) + \lim_{s\to\infty} \underbrace{\int_{0^+}^{\infty} \frac{df}{dt} e^{-st}\, dt}_{\text{THIS IS A FUNCTION } \underline{\text{ONLY OF S}}}$$

$$RHS = f(0^+) - f(0^-)$$

$$LHS = RHS: \quad \lim_{s\to\infty} [sF(s) - f(0^-)] = f(0^+) - f(0^-)$$

$$\therefore \lim_{s\to\infty} sF(s) = f(0^+)$$

Q.E.D.

PROOF OF FVT

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = \underbrace{sF(s) - f(0^-)}_{\text{ALREADY PROVED THIS}} = \underbrace{\int_{0^-}^{\infty} \frac{df}{dt} e^{-st} dt}_{\text{DEFINITION}}$$

NOW TAKE $\lim_{s \to 0}$ OF BOTH SIDES

$$\lim_{s \to 0} \left[sF(s) - f(0^-) \right] = \underbrace{\lim_{s \to 0} \int_{0^-}^{\infty} \frac{df}{dt} e^{-st} dt}$$

$$\parallel$$

$$\int_{0^-}^{\infty} \frac{df(t)}{dt} dt$$

$$\parallel$$

$$\int_{0^-}^{\infty} \frac{df(x)}{dx} dx$$

$$\parallel$$

$$\lim_{t \to \infty} \int_{0^-}^{t} \frac{df(x)}{dx} dx = \lim_{t \to \infty} f(x) \Big|_{0^-}^{t}$$

$$RHS = \lim_{t \to \infty} \left[f(t) - f(0^-) \right]$$

$LHS = RHS$:

CANCELLATION

$$\lim_{s \to 0} \left[sF(s) - \cancel{f(0^-)} \right] = \lim_{t \to \infty} \left[f(t) - \cancel{f(0^-)} \right]$$

$$\therefore \lim_{s \to 0} sF(s) = \lim_{t \to \infty} f(t)$$

Q.E.D.

THEOREM USEFUL ONLY IF $f(\infty)$ EXISTS

WHEN DOES $f(\infty)$ EXIST?

ANS: WHEN POLES ARE IN LHP*

WHY?

*WITH EXCEPTION OF SIMPLE POLE AT THE ORIGIN

EARLIER EXAMPLE

$$F(s) = \frac{4(s+5)(s+10)}{(s+1)(s+2)(s+3)(s+4)} = \frac{K_1}{s+1} + \frac{K_2}{s+2} + \frac{K_3}{s+3} + \frac{K_4}{s+4}$$

$$\left.\frac{4(s+5)(s+10)}{(s+2)(s+3)(s+4)}\right|_{s=-1} = K_1 = 24$$

$$\left.\frac{4(s+5)(s+10)}{(s+1)(s+3)(s+4)}\right|_{s=-2} = K_2 = -48$$

$$\left.\frac{4(s+5)(s+10)}{(s+1)(s+2)(s+4)}\right|_{s=-3} = K_3 = 28$$

$$\left.\frac{4(s+5)(s+10)}{(s+1)(s+2)(s+3)}\right|_{s=-4} = K_4 = -4$$

$$F(s) = \frac{24}{s+1} - \frac{48}{s+2} + \frac{28}{s+3} - \frac{4}{s+4}$$

CHECK VALIDITY

$$\frac{4(s+5)(s+10)}{(s+1)(s+2)(s+3)(s+4)} = \frac{24}{s+1} - \frac{48}{s+2} + \frac{28}{s+3} - \frac{4}{s+4}$$

TRY $s=0$

$$\frac{25}{3} = 24 - \frac{48}{2} + \frac{28}{3} - 1 = \frac{25}{3} \checkmark$$

$$f(t) = (24e^{-t} - 48e^{-2t} + 28e^{-3t} - 4e^{-4t})u(t)$$

$$\text{FVT: } \lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{s\,4(s+5)(s+10)}{(s+1)(s+2)(s+3)(s+4)}$$

$$= 0 \checkmark$$

$$\lim_{t\to\infty} f(t) = 0$$

$$\text{IVT: } \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{s\,4(s+5)(s+10)}{(s+1)(s+2)(s+3)(s+4)}$$

$$= 0$$

WHAT IS $F(0^+)$?

$$(24 - 48 + 28 - 4)\underbrace{a(0^+)}_{\text{WHAT IS THIS ?}}$$

Practical Perspective

Transient Effects

The circuit introduced in the Practical Perspective at the beginning of the chapter is repeated in Fig. 12.18 with the switch closed and the chosen sinusoidal source.

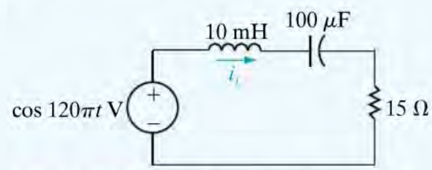


Figure 12.18 ▲ A series *RLC* circuit with a 60 Hz sinusoidal source.

We use the Laplace methods to determine the complete response of the inductor current, $i_L(t)$. To begin, use KVL to sum the voltages drops around the circuit, in the clockwise direction:

$$15i_L(t) + 0.01\frac{di_L(t)}{dt} + \frac{1}{100 \times 10^{-6}}\int_0^t i_L(x)dx = \cos 120\pi t \quad (12.104)$$

Now we take the Laplace transform of Eq. 12.104, using Tables 12.1 and 12.2:

$$15I_L(s) + 0.01sI_L(s) + 10^4\frac{I_L(s)}{s} = \frac{s}{s^2 + (120\pi)^2} \quad (12.105)$$

Next, rearrange the terms in Eq. 12.105 to get an expression for $I_L(s)$:

$$I_L(s) = \frac{100s^2}{\left[s^2 + 1500s + 10^6\right]\left[s^2 + (120\pi^2)\right]} \quad (12.106)$$

Note that the expression for $I_L(s)$ has two complex conjugate pairs of poles, so the partial fraction expansion of $I_L(s)$ will have four terms:

$$I_L(s) = \frac{K_1}{(s + 750 - j661.44)} + \frac{K_1^*}{(s + 750 + j661.44)} + \frac{K_2}{(s - j120\pi)} + \frac{K_2^*}{(s + j120\pi)} \quad (12.107)$$

Determine the values of K_1 and K_2:

$$K_1 = \left.\frac{100s^2}{\left[s + 750s + j661.44\right]\left[s^2 + (120\pi)^2\right]}\right|_{s=-750+j661.44} = 0.07357\angle -97.89^\circ$$

$$K_2 = \left.\frac{100s^2}{\left[s^2 + 1500s + 10^6\right]\left[s + j120\pi\right]}\right|_{s=j120\pi} = 0.018345\angle 56.61^\circ \quad (12.108)$$

Finally, we can use Table 12.3 to calculate the inverse Laplace transform of Eq. 12.107 to give $i_L(t)$:

$$i_L(t) = 147.14e^{-750t}\cos(661.44t - 97.89^\circ) + 36.69\cos(120\pi t + 56.61^\circ)\text{ mA} \quad (12.109)$$

The first term of Eq. 12.109 is the transient response, which will decay to essentially zero in about 7 ms. The second term of Eq. 12.109 is the steady-state response, which has the same frequency as the 60 Hz sinusoidal source and will persist so long as this source is connected in the circuit. Note that the amplitude of the steady-state response is 36.69 mA, which is less than the 40 mA current rating of the inductor. But the transient response has an initial amplitude of 147.14 mA, far greater than the 40 mA current rating. Calculate the value of the inductor current at $t = 0$:

$$i_L(0) = 147.14(1)\cos(-97.89°) + 36.69\cos(56.61°) = -6.21\mu\text{A}$$

Clearly, the transient part of the response does not cause the inductor current to exceed its rating initially. But we need a plot of the complete response to determine whether or not the current rating is ever exceeded, as shown in Fig. 12.19. The plot suggests we check the value of the inductor current at 1 ms:

$$i_L(0.001) = 147.14e^{-0.75}\cos(-59.99°) + 36.69\cos(78.21°) = 42.4\text{ mA}$$

Thus, the current rating is exceeded in the inductor, at least momentarily. If we determine that we never want to exceed the current rating, we should reduce the magnitude of the sinusoidal source. This example illustrates the importance of considering the complete response of a circuit to a sinusoidal input, even if we are satisfied with the steady-state response.

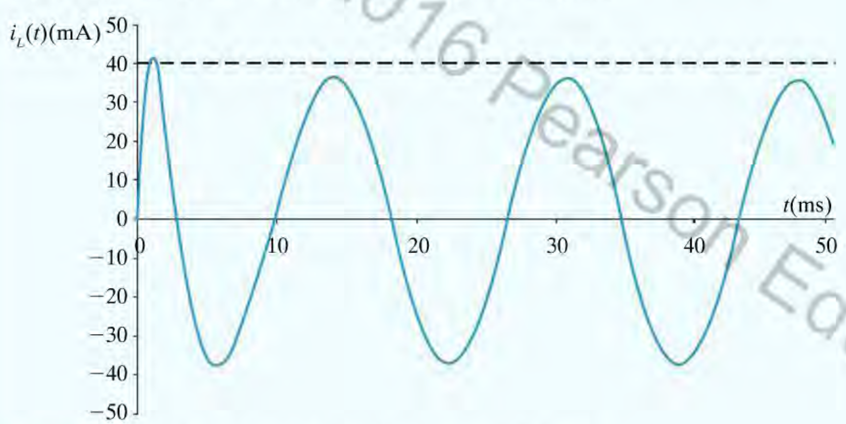


Figure 12.19 ▲ Plot of the inductor current for the circuit in Fig. 12.18.

NOTE: Access your understanding of the Practical Perspective by trying Chapter Problems 12.55 and 12.56.

Summary

- The **Laplace transform** is a tool for converting time-domain equations into frequency-domain equations, according to the following general definition:

$$\mathscr{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt = F(s),$$

where $f(t)$ is the time-domain expression, and $F(s)$ is the frequency-domain expression. (See page 428.)

- The **step function** $Ku(t)$ describes a function that experiences a discontinuity from one constant level to another at some point in time. K is the magnitude of the jump; if $K = 1$, $Ku(t)$ is the **unit step function**. (See page 429.)
- The **impulse function** $K\delta(t)$ is defined

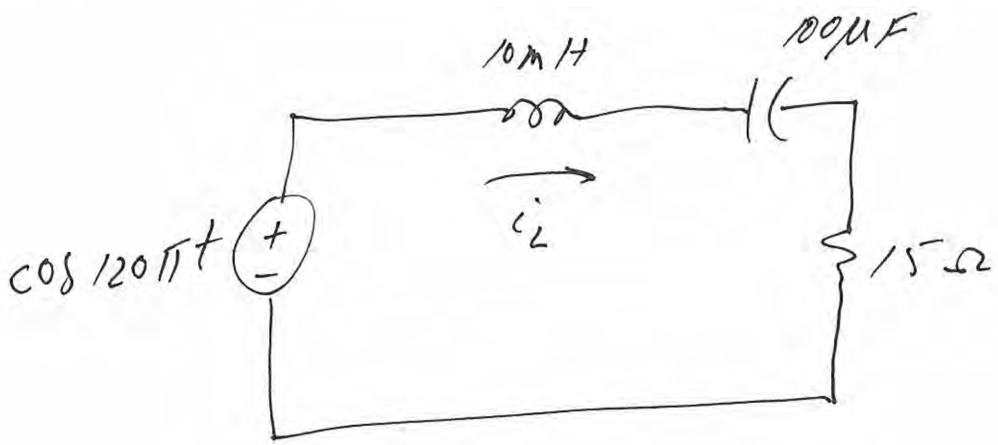
$$\int_{-\infty}^{\infty} K\delta(t)dt = K,$$

$$\delta(t) = 0, \quad t \neq 0.$$

K is the strength of the impulse; if $K = 1$, $K\delta(t)$ is the **unit impulse function**. (See page 431.)

- A **functional transform** is the Laplace transform of a specific function. Important functional transform pairs are summarized in Table 12.1. (See page 434.)
- **Operational transforms** define the general mathematical properties of the Laplace transform. Important operational transform pairs are summarized in Table 12.2. (See page 435.)
- In linear lumped-parameter circuits, $F(s)$ is a rational function of s. (See page 442.)
- If $F(s)$ is a proper rational function, the inverse transform is found by a partial fraction expansion. (See page 442.)
- If $F(s)$ is an improper rational function, it can be inverse-transformed by first expanding it into a sum of a polynomial and a proper rational function. (See page 451.)

PRACTICAL PERSPECTIVES PG ~~458~~ 456 10^{TH} ED



$$\omega = 120\pi$$
$$= 60(2\pi)$$

NO STORED ENERGY

MESH: $$15 i_L(t) + 0.01 \frac{d i_L(t)}{dt} + \frac{1}{100 \times 10^{-6}} \int_0^t i_L(x)\,dx$$
$$= \cos 120\pi t$$

FREQUENCY DOMAIN:

$$15 I_L(s) + 0.01 s I_L(s) + 10^4 \frac{I_L(s)}{s} = \frac{s}{s^2 + (120\pi)^2}$$

$$I_L(s) = \frac{\left(\dfrac{s}{s^2 + (120\pi)^2}\right)}{0.01 s + 15 + 10^4/s}$$

$$= \frac{100 s^2}{\left[s^2 + (120\pi)^2\right]\left[s^2 + 1500 s + 10^6\right]}$$

DOES $I_L(s)$ HAVE PROPER UNITS?

$$R i_L(t) + L \frac{d i_L(t)}{dt} + \frac{1}{C}\int_0^t i_L(x)\,dx = v_s(t)$$

$$v_s = \cos 120\pi t \quad V$$

$$V_s = \frac{s}{s^2 + (120\pi)^2} \quad V \cdot SEC$$

$\vdots$

UNITS = $\frac{\text{VOLTS}}{\text{HENRYS}}$

$$I_L(s) = \frac{100\, s^2}{[s^2 + (120\pi)^2][s^2 + 1500s + 10^6]}$$

$$[I_L(s)] = \frac{\text{VOLTS}}{\text{HENRYS}} \frac{1}{[s^2]} = \frac{\text{VOLTS} \cdot \text{SEC}^2}{\text{VOLT} \cdot \text{SEC}/A}$$

$$= A \cdot SEC \quad \checkmark$$

FACTORING TERMS IN DENOMINATOR

$$s^2 + (120\pi)^2 = (s + j120\pi)(s - j120\pi)$$

$$s^2 + 1500s + 10^6 = (s + 750 - j661.44)(s + 750 + j661.44)$$

$$I_L(s) = \frac{K_1}{s + 750 - j661.44} + \frac{K_1^*}{s + 750 + j661.44}$$

$$+ \frac{K_2}{s - j120\pi} + \frac{K_2^*}{s + j120\pi}$$

DISTINCT ROOTS

⋮

(NOTE TYPO IN EQ 12.108)

$$K_1 = 0.07357 \angle 97.89^\circ$$

$$K_2 = 0.018345 \angle 56.61^\circ$$

$$I_L(s) = \frac{0.07357\angle 97.89^\circ}{s + 750 - j661.44} + \frac{0.07357\angle -97.89^\circ}{s + 750 + j661.44}$$

$$+ \frac{0.018345\angle 56.61^\circ}{s - j120\pi} + \frac{0.018345\angle -56.61^\circ}{s + j120\pi}$$

FROM TABLE 12.3

$$\frac{K}{s+\alpha-j\beta} + \frac{K^*}{s+\alpha+j\beta} \longleftrightarrow 2|K|e^{-\alpha t}\cos(\beta t+\theta)u(t)$$

$$i(t) = \Big[147.14e^{-750t}\cos(661.44t - 97.89^\circ)$$

$$+\,36.69\cos(120\pi t + 56.61^\circ)\Big]u(t)\ \mathrm{mA}$$

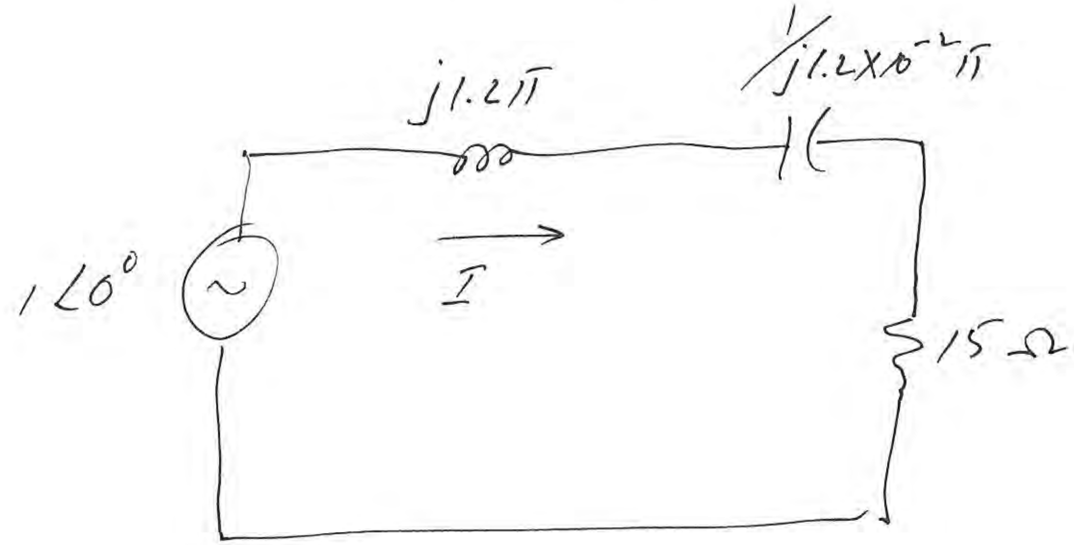
$$661.44 = 2\pi/T \longrightarrow T = 9.5\ \mathrm{ms}$$

LENGTH OF TRANSIENT

$$e^{-5} = 0.67\ \%$$

$$750t = 5 \longrightarrow t = 6.67\ \mathrm{ms}$$

STEADY-STATE ANALYSIS



MESH EQ: $\underline{I}\left(j1.2\pi + \frac{1}{j1.2\times10^{-2}\pi} + 15\right) = 1\angle 0^\circ$

$$\underline{I} = \frac{1\angle 0^\circ}{15 + j1.2\pi + \frac{1}{j1.2\times10^{-2}\pi}}$$

$$\underline{I} = \frac{1.2\times10^{-2}\pi}{0.18\pi + j(0.0144\pi^2 - 1)}$$

$$= \frac{1.2\times10^{-2}\pi}{0.18\pi - j0.858} = \frac{1.2\times10^{-2}\pi}{1.03\angle -56.6^\circ}$$

$$= 3.66\times10^{-2}\angle 56.6^\circ$$

$$i(t) = 36.6\cos(120\pi t + 56.6^\circ)\ \text{mA}$$

RETURN TO FREQ DOMAIN EXPRESSION

$$I_L(s) = \frac{100s^2}{(s^2+1500s+10^6)(s^2+(120\pi)^2)}$$

APPLY INITIAL VALUE THEOREM:

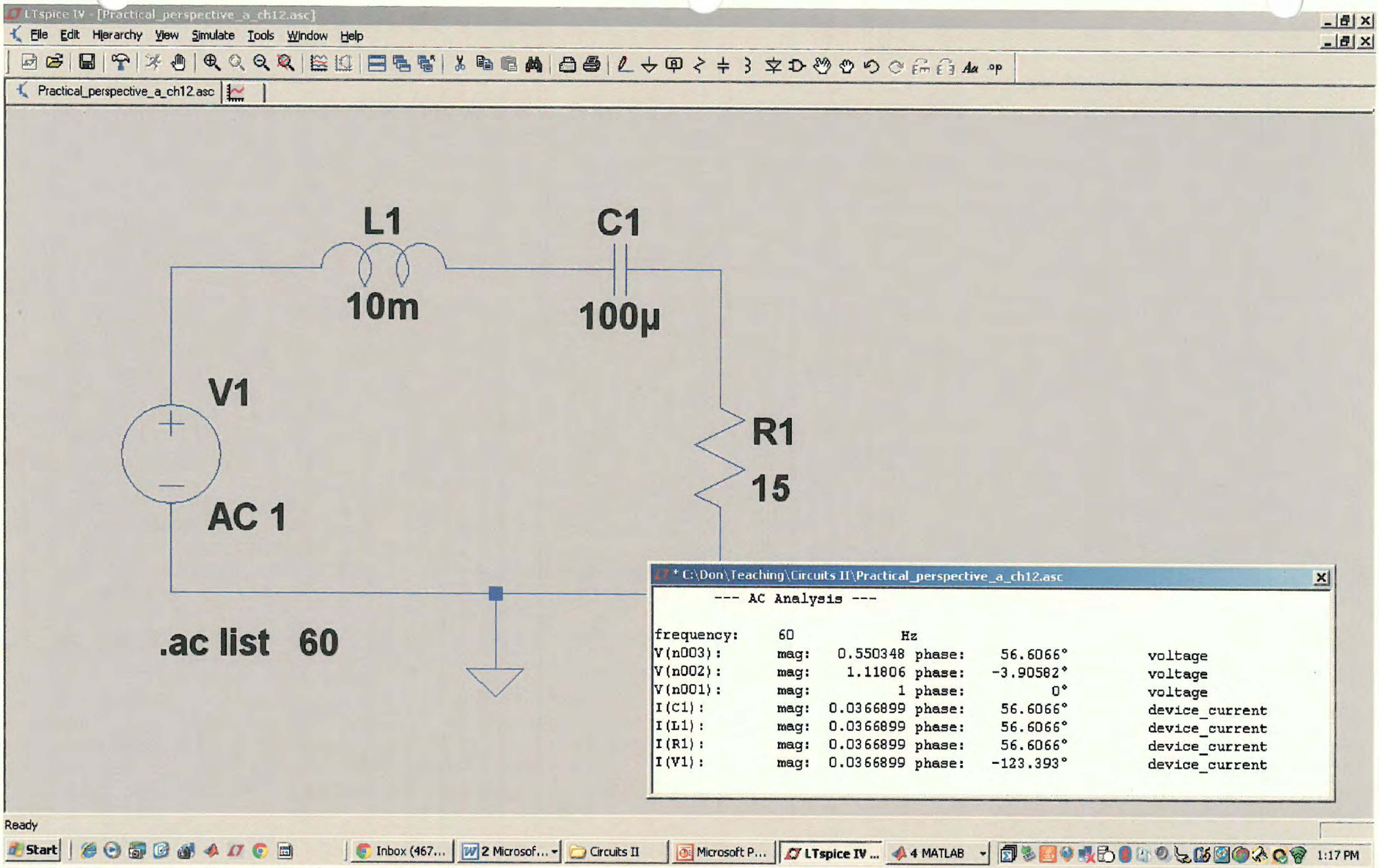
$$\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)$$

$$i_L(0^+) = \lim_{s \to \infty} \frac{100s^3}{(s^2+1500s+10^6)(s^2+(120\pi)^2)}$$

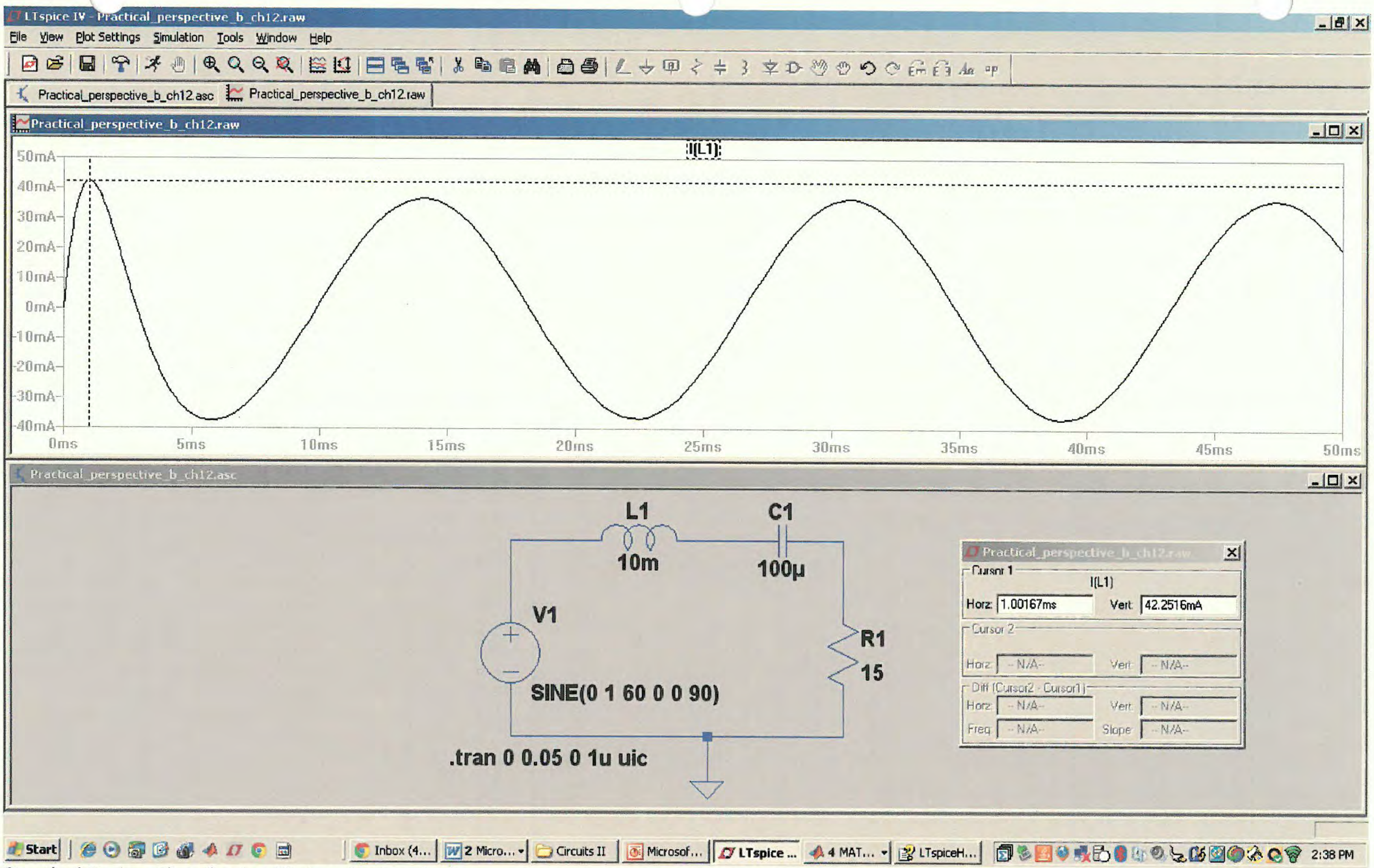
$$= \underline{\underline{0}}$$

WHERE DID TEXT GO WRONG

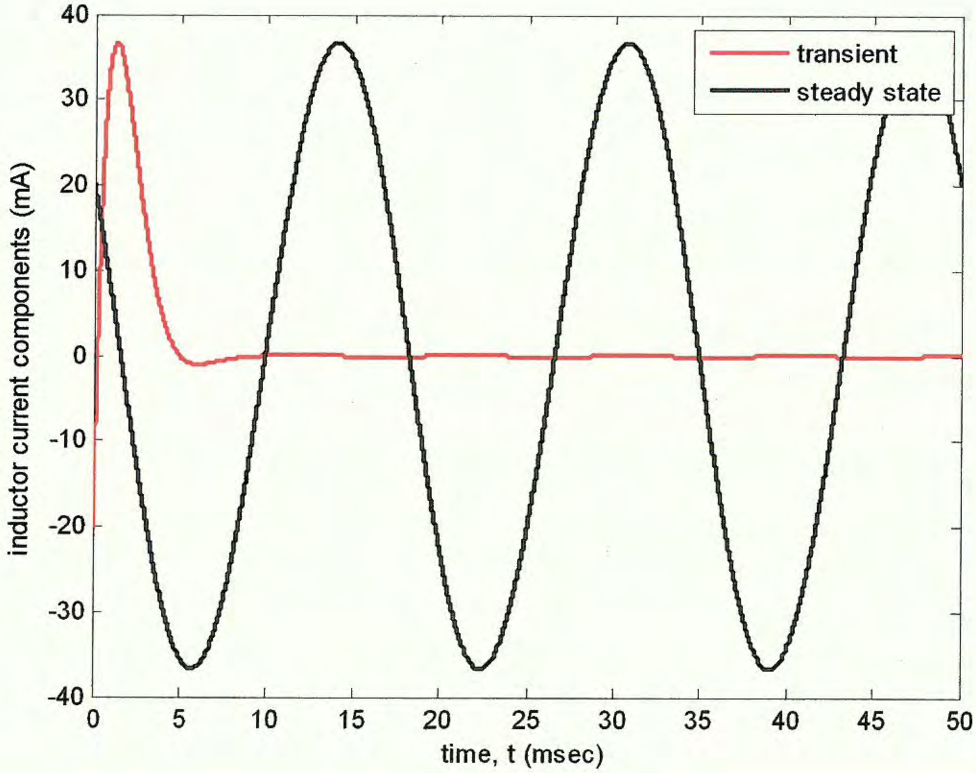
IN CALCULATING $i_L(0)$?



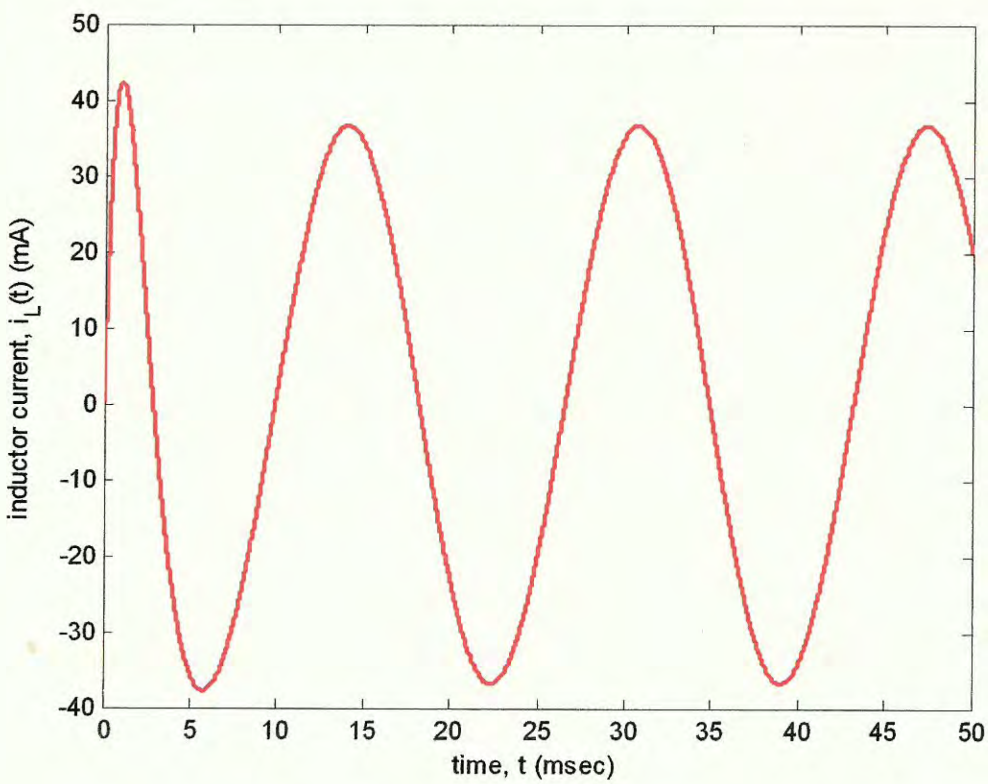
Practical Perspective chapter 12, steady state analysis



Practical Perspective chapter 12, transient analysis



transient
steady state
inductor current components (mA)
time, t (msec)
40
30
20
10
0
-10
-20
-30
-40
0
5
10
15
20
25
30
35
40
45
50



inductor current, $i_L(t)$ (mA)
time, t (msec)
50
40
30
20
10
0
-10
-20
-30
-40
0
5
10
15
20
25
30
35
40
45
50

```matlab
% practical_perspective_ch_12
% 02/23/14 D D Duncan
%
N = 10000;
t = linspace(0,0.05,N);
cdtr = pi/180;
alpha = 750;
beta = sqrt(1e6-(750)^2);
s_r = -alpha+j*beta;
omega = 120*pi;
K1 = (100*s_r^2)/((s_r+alpha + j*beta)*(s_r^2+omega^2));
s_r = j*omega;
K2 = (100*s_r^2)/((s_r^2+2*alpha*s_r + 1e6)*(s_r+j*omega));
i_tran_0 = 2*abs(K1);
phi_tran = atan2(imag(K1),real(K1));
i_ss_0 = 2*abs(K2);
phi_ss = atan2(imag(K2),real(K2));
i_L_tran = i_tran_0*exp(-alpha*t).*cos(beta*t + phi_tran);
i_L_ss =  i_ss_0*cos(omega*t + phi_ss);
i_L = i_L_tran + i_L_ss;
figure(3);plot(t*1000,i_L_tran*1000,'r-',t*1000,i_L_ss*1000,'k-');
legend('transient','steady state');
xlabel('time, t (msec)');ylabel('inductor current components (mA)')
figure(4);plot(t*1000,i_L*1000,'r-');
xlabel('time, t (msec)');ylabel('inductor current, i_L(t) (mA)')
% check value at t = 0;
initial_curret = i_L(1)
```